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Type-2 fuzzy subalgebras of BCK-algebras and BCI-algebras

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Abstract. Using the type-2 fuzzy set theory, we study the subalgebra of BCK/BCI-algebras. We introduce the concept of (strong) type-2 fuzzy subalgebras of BCK/BCI-algebras and investigate several properties. We establish the relationship between the fuzzy subalgebra, the strong type-2 fuzzy subalgebra and the type-2 fuzzy subalgebra. We find a way to derive the (strong) type-2 fuzzy subalgebra from the fuzzy subalgebra and vice versa. Given a type-2 fuzzy set, we create the slice and the cut, and find the conditions for the slice to be a fuzzy subalgebra. We use the cut to explore the characterization of a strong type-2 fuzzy subalgebra. We introduce the support and the core for the slice and explore the conditions under which they become subalgebras. We introduce the super level set and the infer level set for type-2 fuzzy set and explore the conditions under which they become subalgebras.

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1. Introduction

Type-2 fuzzy sets were introduced by Zadeh [14] in 1975 as an extension of the ordinary (type-1) fuzzy sets, designed to better handle uncertainty and imprecision in membership functions. Later, the idea was greatly developed by Mendel and others, especially for interval type-2 fuzzy sets used in practical applications such as fuzzy logic controllers (See [7, 9, 10, 11, 13]). Type-2 fuzzy sets have a wide range of applications in various fields, including decision-making under uncertainty, signal & image processing, control systems, computing using words, and human-centric systems.

In this paper, we use the type-2 fuzzy set theory to study the subalgebra of BCK/BCI-algebras. A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers (See [2, 3, 4, 12]). Given a type-2 fuzzy set $\eth$ on a set X and $(t, \beta) \in [0, 1] \times [0, 1]$, we carry out the following tasks.

(1) Introducing the (strong) type-2 fuzzy subalgebra $\eth$ in a BCK/BCI-algebra.
(2) To establish the relationship between the fuzzy subalgebra, the strong type-2 fuzzy subalgebra and the type-2 fuzzy subalgebra.
(3) Finding examples where strong type-2 fuzzy subalgebra does not become type-2 fuzzy subalgebra.
(4) Finding the conditions under which type-2 fuzzy subalgebra (resp., fuzzy subalgebra) can be derived from fuzzy subalgebra (resp., type-2 fuzzy subalgebra).
(5) To establish the t-slice $\eth_t$ and the (t, β)-cut $\eth^t_\beta$ of $\eth$.
(6) Finding conditions for the $\eth^t_\beta$ (resp., $\eth_t$) to be a subalgebra (resp., fuzzy subalgebra).
(7) Using the t-slice to find the conditions under which $\eth$ can be a (strong) type-2 fuzzy subalgebra.
(8) To establish the t-support and the t-core of the t-slice.
(9) Exploring the conditions under which t-support and t-core will be subalgebras.
(10) To establish the super and infer β-level sets, and finding conditions under which they become subalgebras.

2. Preliminaries

In this section we will provide some formal definition and properties of BCK/BCI-algebras and fuzzy concepts that will be used throughout the paper.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I_1) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ (((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,
(I_2) $(\forall \mathfrak{a}, \mathfrak{b} \in X)\ ((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,
(I_3) $(\forall \mathfrak{a} \in X)\ (\mathfrak{a} * \mathfrak{a} = 0)$,
(I_4) $(\forall \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0 \ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall \mathfrak{a} \in X)\ (0 * \mathfrak{a} = 0)$,

then X is called a *BCK-algebra*.

The order relation $\leq_X$ in a BCK/BCI-algebra $\mathcal{X}$ is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq_X \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0).$$

Every BCK/BCI-algebra X satisfies the following conditions (See [2, 12]):

$$(\forall \mathfrak{a} \in X)\,(\mathfrak{a} * 0 = \mathfrak{a})\,, \tag{2.1}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} \leq_X \mathfrak{b} \Rightarrow \mathfrak{a} * \mathfrak{c} \leq_X \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq_X \mathfrak{c} * \mathfrak{a})\,, \tag{2.2}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,. \tag{2.3}$$

A subset F of a BCK/BCI-algebra X is called a *subalgebra* of X (See [2, 12]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in F)(\mathfrak{a} * \mathfrak{b} \in F). \tag{2.4}$$

For more information on BCI-algebra and BCK-algebra, please refer to the books [2, 12].

A fuzzy set f in X is called a *fuzzy subalgebra* of a BCK/BCI-algebra $\mathcal{X} := (X, *, 0)$ (See [5, 6]) if it satisfies

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(f(\mathfrak{a} * \mathfrak{b}) \geq \min\{f(\mathfrak{a}), f(\mathfrak{b})\}).$$

A *type-2 fuzzy set*, denoted by A, on the nonempty universe X is given by

$$A = \{((\mathfrak{a}, u), \eth_A(\mathfrak{a}, t)) \mid \mathfrak{a} \in X, t \in J_{\mathfrak{a}} \subseteq [0, 1]\}$$

in which $0 \leq \eth_A(\mathfrak{a}, t) \leq 1$, where t is the *primary membership*, and $\eth_A(\mathfrak{a}, t)$ is the *secondary membership* of A (see [1, 10]).

In general, a type-2 fuzzy set on the nonempty universe X is characterized by a mapping

$$\eth : X \times [0, 1] \to [0, 1],$$

where for each $\mathfrak{a} \in X$, $\eth(\mathfrak{a}, t)$ provides the degree of membership of t in the fuzzy membership of $\mathfrak{a}$, that is, a type-2 fuzzy set is a fuzzy set whose membership values are themselves fuzzy.

3. Type-2 fuzzy subalgebras

In what follows, let $\mathcal{X} := (X, *, 0)$ (or simply $\mathcal{X}$) be a BCK-algebra or a BCI-algebra unless otherwise specified.

Definition 3.1. A type-2 fuzzy set $\eth$ on X is called a *type-2 fuzzy subalgebra* of $\mathcal{X}$, if it satisfies

$$\begin{array}{l} (\forall x, y \in X)(\forall (t_x, t_y) \in [0, 1] \times [0, 1]) \\ (\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\}). \end{array} \tag{3.1}$$

Remark 3.2. In the type-2 fuzzy subalgebra $\eth$ of $\mathcal{X}$, the primary memberships t_x and t_y are generally not the same.

A type-2 fuzzy set $\eth$ on X is called a *strong type-2 fuzzy subalgebra* of $\mathcal{X}$, if $t_x = t_y$ in (3.1). Hence a type-2 fuzzy set $\eth$ on X is a strong type-2 fuzzy subalgebra of $\mathcal{X}$ if and only if it satisfies

$$\begin{array}{l} (\forall x, y \in X)(\forall (t \in [0, 1]) \\ (\eth(x * y, t) \geq \min\{\eth(x, t), \eth(y, t)\}). \end{array} \tag{3.2}$$

Example 3.3. Consider a BCK-algebra $\mathcal{X} := (X, *, \varrho_0)$, where $X = \{\varrho_0, \varrho_1\}$ and the binary operation $*$ is given by Table 1. Define a fuzzy set $f : X \to [0, 1]$ by $f(\varrho_0) = 1$ and $f(\varrho_1) = t \in [0, 1)$. Let $\eth$ be a type-2 fuzzy set on X given as follows:

$$\eth : X \times [0, 1] \to [0, 1], \ (x, s) \mapsto \begin{cases} 1 & \text{if } s \leq f(x), \\ 0 & \text{if } s > f(x). \end{cases}$$

Table 1. $*$-table

$*$	ϱ_0	ϱ_1
ϱ_0	ϱ_0	ϱ_0
ϱ_1	ϱ_1	ϱ_0

For every $(t_x, t_y) \in [0,1] \times [0,1]$, if $(x, y) = (\varrho_0, \varrho_0)$, then

$$\begin{aligned}\eth(x * y, \min\{t_x, t_y\}) &= \eth(\varrho_0, \min\{t_x, t_y\}) = 1 = \min\{1, 1\} \\ &\geq \min\{\eth(\varrho_0, t_x), \eth(\varrho_0, t_y)\} = \min\{\eth(x, t_x), \eth(y, t_y)\}.\end{aligned}$$

If $(x, y) = (\varrho_1, \varrho_1)$, then we should consider two cases:

(i) $t_x \leq t$ and $t_y \leq t$,
(ii) either $t_x > t$ or $t_y > t$.

For the first case, we have $\eth(x, t_x) = \eth(\varrho_1, t_x) = 1 = \eth(\varrho_1, t_y) = \eth(y, t_y)$ and $\eth(x * y, \min\{t_x, t_y\}) = \eth(\varrho_0, \min\{t_x, t_y\}) = 1$. Then

$$\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\}.$$

For the second case, we have $\min\{t_x, t_y\} > t = f(\varrho_1)$ and then

$$\begin{aligned}\eth(x * y, \min\{t_x, t_y\}) &= \eth(\varrho_0, \min\{t_x, t_y\}) = 0 \\ &= \min\{\eth(\varrho_1, t_x), \eth(\varrho_1, t_y)\} = \min\{\eth(x, t_x), \eth(y, t_y)\}.\end{aligned}$$

If $(x, y) = (\varrho_0, \varrho_1)$, then

$$\eth(x * y, \min\{t_x, t_y\}) = \eth(\varrho_0, \min\{t_x, t_y\}) = 1 \geq \min\{\eth(x, t_x), \eth(y, t_y)\}.$$

If $(x, y) = (\varrho_1, \varrho_0)$, then

$$\eth(x * y, \min\{t_x, t_y\}) = \eth(\varrho_1, \min\{t_x, t_y\}) = \begin{cases} 1 & \text{if } \min\{t_x, t_y\} \leq f(\varrho_1) = t, \\ 0 & \text{if } \min\{t_x, t_y\} > f(\varrho_1) = t.\end{cases}$$

The case $\min\{t_x, t_y\} > f(\varrho_1) = t$ forces $t_x > f(\varrho_1) = t$. Then $\eth(x, t_x) = \eth(\varrho_1, t_x) = 0$. Thus $\eth(x * y, \min\{t_x, t_y\}) \geq 0 = \min\{\eth(x, t_x), \eth(y, t_y)\}$. So $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$.

It is clear that every type-2 fuzzy subalgebra is a strong type-2 fuzzy subalgebra, but the converse is not always correct as shown in the following example.

Example 3.4. Let $X = \{\varrho_0, \varrho_1, \varrho_2\}$ be a set with the chain order $\varrho_0 \leq_X \varrho_1 \leq_X \varrho_2$. Define a binary operation $*$ by

$$(\forall x, y \in X) \left(x * y = \begin{cases} \varrho_0 & \text{if } x \leq_X y, \\ x & \text{otherwise.}\end{cases} \right).$$

Then $\mathcal{X} := (X, *, \varrho_0)$ is a BCK-algebra (See [12]). Define a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$ by

$$\begin{cases} \eth(\varrho_0, t) = 0 \;\; \text{for all } t \in [0,1], \\ \eth(\varrho_1, t) = \begin{cases} 1 & \text{if } t \in [0.7, 1], \\ 0 & \text{if } t \in [0, 0.7), \end{cases} \\ \eth(\varrho_2, t) = \begin{cases} 0 & \text{if } t \in (0.5, 1), \\ 1 & \text{if } t \in [0, 0.5]. \end{cases} \end{cases}$$

It is routine to verify that $\eth$ is a strong type-2 fuzzy subalgebra of $\mathcal{X}$. If we take $t_1 \in [0.7, 1]$ and $t_2 \in [0, 0.5]$, then $\eth(\varrho_1, t_1) = 1$ and $\eth(\varrho_2, t_2) = 1$. Thus

$$\eth(\varrho_1 * \varrho_2, \min\{t_1, t_2\}) = \eth(\varrho_0, t_2) = 0 < 1 = \min\{\eth(\varrho_1, t_1), \eth(\varrho_2, t_2)\}.$$

So $\eth$ is not type-2 fuzzy subalgebra of $\mathcal{X}$.

In general, not every fuzzy subalgebra automatically is a type-2 fuzzy subalgebra and vice versa. However, we can see that every fuzzy subalgebra can be realized as a special case of a type-2 fuzzy subalgebra, and every type-2 fuzzy subalgebra induces a fuzzy subalgebra as shown below.

Theorem 3.5. *If f is a fuzzy subalgebra of $\mathcal{X}$, then the type-2 fuzzy set*

$$\eth : X \times [0, 1] \to [0, 1],\ (x, t) \mapsto f(x)$$

is a type-2 fuzzy subalgebra of $\mathcal{X}$.

Proof. For every $x, y \in X$ and $(t_x, t_y) \in [0, 1] \times [0, 1]$, we have

$$\eth(x * y, \min\{t_x, t_y\}) = f(x * y) \geq \min\{f(x), f(y)\} = \min\{\eth(x, t_x), \eth(y, t_y)\}.$$

Then $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$. □

Theorem 3.6. *Let $\eth$ be a type-2 fuzzy subalgebra of $\mathcal{X}$. If we give a fuzzy set f in X by*

$$f : X \to [0, 1],\ x \mapsto \sup_{t \in [0,1]} \eth(x, t),$$

then f is a fuzzy subalgebra of $\mathcal{X}$.

Proof. Assume $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$. Then

$$\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\}$$

for all $x, y \in X$ and $(t_x, t_y) \in [0, 1] \times [0, 1]$. Taking the supremum over t_x, t_y and $\min\{t_x, t_y\}$ yields

$$\sup_{\min\{t_x, t_y\} \in [0,1]} \eth(x * y, \min\{t_x, t_y\}) \geq \sup_{t_x, t_y \in [0,1]} \min\{\eth(x, t_x), \eth(y, t_y)\}.$$

Since t_x and t_y move independently, we have

$$\sup_{t_x, t_y \in [0,1]} \min\{\eth(x, t_x), \eth(y, t_y)\} = \min\left\{ \sup_{t_x \in [0,1]} \eth(x, t_x),\ \sup_{t_y \in [0,1]} \eth(y, t_y) \right\}.$$

Thus it follows that

$$\begin{aligned} f(x * y) &= \sup_{\min\{t_x, t_y\} \in [0,1]} \eth(x * y, \min\{t_x, t_y\}) \\ &\geq \min\left\{ \sup_{t_x \in [0,1]} \eth(x, t_x),\ \sup_{t_y \in [0,1]} \eth(y, t_y) \right\} \\ &= \min\{f(x), f(y)\} \end{aligned}$$

for all $x, y \in X$. So f is a fuzzy subalgebra of $\mathcal{X}$. □

Theorem 3.7. *If $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, then the fuzzy set*

$$f : X \to [0,1],\ x \mapsto \inf_{t\in[0,1]} \eth(x,t),$$

is a fuzzy subalgebra of $\mathcal{X}$.

Proof. Assume $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$. Then

$$\eth(x*y, \min\{t_x, t_y\}) \geq \min\{\eth(x,t_x), \eth(y,t_y)\}$$

for all $x, y \in X$ and $(t_x, t_y) \in [0,1] \times [0,1]$. If we take the infimum over t_x, t_y and $\min\{t_x, t_y\}$ on both sides, then

$$\begin{aligned} f(x*y) &= \inf_{\min\{t_x,t_y\}\in[0,1]} \eth(x*y, \min\{t_x,t_y\}) \\ &\geq \inf_{(t_x,t_y)\in[0,1]\times[0,1]} \min\{\eth(x,t_x), \eth(y,t_y)\} \\ &= \min\{\inf_{t_x\in[0,1]} \eth(x,t_x), \inf_{t_y\in[0,1]} \eth(y,t_y)\} \\ &= \min\{f(x), f(y)\}. \end{aligned}$$

Thus f is a fuzzy subalgebra of $\mathcal{X}$. □

Given a type-2 fuzzy set $\eth$ on X, consider the set

$$\eth^t_\beta := \{x \in X \mid \eth(x,t) \geq \beta\},$$

where $(t,\beta) \in [0,1]\times[0,1]$. We say $\eth^t_\beta$ is the (t,β)-*cut* of $\eth$. It is clear that if $\beta_1 \leq \beta_2$ in $[0,1]$, then $\eth^t_{\beta_2} \subseteq \eth^t_{\beta_1}$.

Given a type-2 fuzzy set $\eth$ on X and $t \in [0,1]$, define a fuzzy set $\eth_t$ on X at t as follows:

$$\eth_t : X \to [0,1],\ x \mapsto \eth(x,t),$$

and it is called the t-*slice* of $\eth$. Then

$$\eth^t_\beta = \{x \in X \mid \eth_t(x) \geq \beta\}$$

which is exactly the β-cut of the t-slice $\eth_t$ of $\eth$.

We find that the t-slice $\eth_t$ of a type-2 fuzzy set $\eth$ is not usually a fuzzy subalgebra. In fact, given a singleton set $X = \{\mathfrak{a}\}$, consider its power set $\mathcal{P}(X) = \{\varnothing, \{\mathfrak{a}\}\}$. It is clear that $(\mathcal{P}(X), \setminus, \varnothing)$ is a BCI-algebra where $\setminus$ is the set difference (See [2]). Define a type-2 fuzzy set $\eth : \mathcal{P}(X) \times [0,1] \to [0,1]$ by

$$\eth(\varnothing, t) = \begin{cases} 0.47 & \text{if } t = 0.6, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \eth(\{\mathfrak{a}\}, t) = \begin{cases} 0.74 & \text{if } t = 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \eth_t(\{\mathfrak{a}\} \setminus \{\mathfrak{a}\}) = \eth_t(\varnothing) = \eth(\varnothing, t) &= 0.47 < 0.74 \\ &= \min\{\eth(\{\mathfrak{a}\}, t), \eth(\{\mathfrak{a}\}, t)\} \\ &= \min\{\eth_t(\{\mathfrak{a}\}), \eth_t(\{\mathfrak{a}\})\}. \end{aligned}$$

Thus $\eth_t$ is not a fuzzy subalgebra of the BCK-algebra $\mathcal{P}(X) = \{\varnothing, \{\mathfrak{a}\}\}$.

Theorem 3.8. *Given a type-2 fuzzy set* $\eth$ *on* X *and* $t \in [0,1]$, *if its* t*-slice* $\eth_t$ *is a fuzzy subalgebra of* $\mathcal{X}$, *then the nonempty* (t,β)*-cut of* $\eth$ *is a subalgebra of* $\mathcal{X}$ *for all* $\beta \in [0,1]$.

Proof. Let $(t,\beta) \in [0,1] \times [0,1]$ be such that $\eth^t_\beta \neq \varnothing$. If $x, y \in \eth^t_\beta$, then $\eth(x,t) \geq \beta$ and $\eth(y,t) \geq \beta$. Thus

$$\eth(x * y, t) = \eth_t(x * y) \geq \min\{\eth_t(x), \eth_t(y)\} = \min\{\eth(x,t), \eth(y,t)\} \geq \beta.$$

So $x * y \in \eth^t_\beta$. Hence $\eth^t_\beta$ is a subalgebra of $\mathcal{X}$. □

Theorem 3.9. *If a type-2 fuzzy set* $\eth$ *on* X *is a type-2 fuzzy subalgebra of* $\mathcal{X}$, *then its* t*-slice is a fuzzy subalgebra of* $\mathcal{X}$ *for all* $t \in [0,1]$.

Proof. If $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, then

$$\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\}$$

for all $x, y \in X$ and $(t_x, t_y) \in [0,1] \times [0,1]$. If we take $t = t_x = t_y$, then

$$\eth_t(x * y) = \eth(x * y, t) \geq \min\{\eth(x,t), \eth(y,t)\} = \min\{\eth_t(x), \eth_t(y)\}.$$

Thus $\eth_t$ is a fuzzy subalgebra of $\mathcal{X}$. □

Corollary 3.10. *If a type-2 fuzzy set* $\eth$ *on* X *is a type-2 fuzzy subalgebra of* $\mathcal{X}$, *then the nonempty* (t,β)*-cut of* $\eth$ *is a subalgebra of* $\mathcal{X}$ *for all* $\beta \in [0,1]$.

The example below shows that the converse of Theorem 3.9 may not be true.

Example 3.11. Consider a BCK-algebra $\mathcal{X} := (X, *, \varrho_0)$, where $X = \{\varrho_0, \varrho_1\}$ and the binary operation $*$ is defined by $\varrho_1 * \varrho_0 = \varrho_1$ and $\varrho_0 * \varrho_0 = \varrho_0 * \varrho_1 = \varrho_1 * \varrho_1 = \varrho_0$. Define a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$ by

$$\eth(\varrho_0, t) = 1 \text{ for all } t \in [0,1]$$

and

$$\eth(\varrho_1, t) = \begin{cases} 1 & \text{if } t \in (0.5, 1], \\ 0 & \text{if } t \in [0, 0.5]. \end{cases}$$

If $t \in (0.5, 1]$, then $\eth_t(\varrho_0) = \eth(\varrho_0, t) = 1 = \eth(\varrho_1, t) = \eth_t(\varrho_1)$, so $\eth_t(x * y) = 1 \geq \min\{\eth_t(x), \eth_t(y)\}$ for all $x, y \in X$. If $t \in [0, 0.5]$, then $\eth_t(\varrho_0) = \eth(\varrho_0, t) = 1$ and $\eth_t(\varrho_1) = \eth(\varrho_1, t) = 0$. Thus we have to consider

$$(x,y) \in \{(\varrho_0, \varrho_0), (\varrho_0, \varrho_1), (\varrho_1, \varrho_0), (\varrho_1, \varrho_1)\}.$$

If $(x,y) = (\varrho_0, \varrho_0)$, then $\eth_t(\varrho_0 * \varrho_0) = \eth_t(\varrho_0) = 1 = \min\{\eth_t(\varrho_0), \eth_t(\varrho_0)\}$. If $(x,y) = (\varrho_0, \varrho_1)$, then $\eth_t(\varrho_0 * \varrho_1) = \eth_t(\varrho_0) = 1 \geq 0 = \min\{\eth_t(\varrho_0), \eth_t(\varrho_1)\}$. If $(x,y) = (\varrho_1, \varrho_0)$, then $\eth_t(\varrho_1 * \varrho_0) = \eth_t(\varrho_1) = 0 = \min\{\eth_t(\varrho_1), \eth_t(\varrho_0)\}$. If $(x,y) = (\varrho_1, \varrho_1)$, then $\eth_t(\varrho_1 * \varrho_1) = \eth_t(\varrho_0) = 1 \geq 0 = \min\{\eth_t(\varrho_1), \eth_t(\varrho_1)\}$. Thus $\eth_t(x * y) = 1 \geq \min\{\eth_t(x), \eth_t(y)\}$ for all $x, y \in X$, that is, $\eth_t$, $t \in [0,1]$, is a fuzzy subalgebra of $\mathcal{X}$. Since

$$\begin{aligned} \eth(\varrho_1 * \varrho_0, \min\{0.7, 0.4\}) &= \eth(\varrho_1, 0.4) = 0 < 1 = \min\{1, 1\} \\ &= \min\{\eth(\varrho_1, 0.7), \eth(\varrho_0, 0.4)\}, \end{aligned}$$

$\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

When will the converse of Theorem 3.9 be established? There is one answer below.

Theorem 3.12. *Let* $\eth$ *be a type-2 fuzzy set on* X *that satisfies*

$$(\forall x \in X)(\forall (t_1, t_2) \in [0,1] \times [0,1])(t_1 \leq t_2 \;\Rightarrow\; \eth(x, t_2) \leq \eth(x, t_1)). \tag{3.3}$$

If the t*-slice* $\eth_t$ *of* $\eth$ *is a fuzzy subalgebra of* X *for all* $t \in [0,1]$*, then* $\eth$ *is a type-2 fuzzy subalgebra of* $\mathcal{X}$*.*

Proof. Suppose $\eth$ satisfies the condition (3.3) and the t-slice $\eth_t$ of $\eth$ is a fuzzy subalgebra of X for all $t \in [0,1]$. For every $(t_x, t_y) \in [0,1] \times [0,1]$, if we put $t := \min\{t_x, t_y\}$, then $t_x \geq t$ and $t_y \geq t$. Thus $\eth_{t_x}(x) = \eth(x, t_x) \leq \eth(x, t) = \eth_t(x)$ and $\eth_{t_y}(y) = \eth(y, t_y) \leq \eth(y, t) = \eth_t(y)$ by (3.3). So it follows that

$$\begin{aligned}\eth(x * y, \min\{t_x, t_y\}) &= \eth(x * y, t) = \eth_t(x * y) \geq \min\{\eth_t(x), \eth_t(y)\} \\ &\geq \min\{\eth_{t_x}(x), \eth_{t_y}(y)\} = \min\{\eth(x, t_x), \eth(y, t_y)\}\end{aligned}$$

for all $x, y \in X$. Hence $\eth$ is type-2 fuzzy subalgebra of $\mathcal{X}$. □

Theorem 3.13. *Let* $\eth$ *be a type-2 fuzzy set on* X*. If the* t*-slice* $\eth_t$ *of* $\eth$ *is a fuzzy subalgebra of* X *for all* $t \in [0,1]$*, then* $\eth$ *is a strong type-2 fuzzy subalgebra of* $\mathcal{X}$*.*

Proof. For every $x, y \in X$ and $t \in [0,1]$, we have

$$\eth(x * y, t) = \eth_t(x * y) \geq \min\{\eth_t(x), \eth_t(y)\} = \min\{\eth(x, t), \eth(y, t)\}.$$

Then $\eth$ is a strong type-2 fuzzy subalgebra of $\mathcal{X}$. □

Theorem 3.14. *A type-2 fuzzy set* $\eth$ *on* X *is a strong type-2 fuzzy subalgebra of* $\mathcal{X}$ *if and only if its nonempty* (t, β)*-cut is a subalgebra of* $\mathcal{X}$ *for all* $(t, \beta) \in [0,1] \times [0,1]$*.*

Proof. Assume that $\eth$ is a strong type-2 fuzzy subalgebra of $\mathcal{X}$. Let $(t, \beta) \in [0,1] \times [0,1]$ be such that $\eth^t_\beta \neq \varnothing$. If $x, y \in \eth^t_\beta$, then $\eth(x, t) \geq \beta$ and $\eth(y, t) \geq \beta$. Thus

$$\eth(x * y, t) \geq \min\{\eth(x, t), \eth(y, t)\} \geq \beta.$$

So $x * y \in \eth^t_\beta$. Hence $\eth^t_\beta$ is a subalgebra of $\mathcal{X}$.

Conversely, suppose the nonempty (t, β)-cut of $\eth$ is a subalgebra of $\mathcal{X}$ for all $(t, \beta) \in [0,1] \times [0,1]$. Put $\beta = \min\{\eth(x, t), \eth(y, t)\}$. Then $x, y \in \eth^t_\beta$. Thus $x * y \in \eth^t_\beta$ since $\eth^t_\beta$ is a subalgebra of $\mathcal{X}$. So

$$\eth(x * y, t) \geq \beta = \min\{\eth(x, t), \eth(y, t)\}.$$

Hence $\eth$ is a strong type-2 fuzzy subalgebra of $\mathcal{X}$. □

Corollary 3.15. *If a type-2 fuzzy set* $\eth$ *on* X *is a type-2 fuzzy subalgebra of* $\mathcal{X}$*, then its nonempty* (t, β)*-cut is a subalgebra of* $\mathcal{X}$ *for all* $(t, \beta) \in [0,1] \times [0,1]$*.*

The example below shows that the converse of Corollary 3.15 is not true in general.

Example 3.16. Consider a BCK/BCI-algebra $\mathcal{X} := (X, *, 0)$, where $X = \mathbb{N} \cup \{0\}$ and $*$ is defined by $x * y = \max\{0, x - y\}$ for all $x, y \in \mathbb{N}$. Define a type-2 fuzzy set $\eth$ on X by

$$\eth : X \times [0,1] \to [0,1],\; (x, t) \mapsto \begin{cases} 1 & \text{if } x = 0 \text{ and } t \in [0,1], \\ 1 & \text{if } x = 7 \text{ and } t \in [0.5,1], \\ 1 & \text{if } x = 4 \text{ and } t \in [0,0.5), \\ 0 & \text{otherwise.} \end{cases}$$

For each fixed t, the $(t,1)$-cut $\eth_1^t = \{x \in X \mid \eth(x,t) \geq 1\}$ is one of the subsets $\{0\}$, $\{0,4\}$, and $\{0,7\}$. If $\beta = 0$, then $\eth_0^t = X$. For the case $\beta \in (0,1]$, we have $\eth_\beta^t = \eth_1^t$. Thus the nonempty (t,β)-cut of $\eth$ is one of $\{0\}$, $\{0,4\}$, $\{0,7\}$ and X, and each of them is a subalgebra of $\mathcal{X}$. If we take $x = 7$, $y = 4$, $t_1 = 0.83$ and $t_2 = 0.38$, then $x * y = 7 * 4 = \max\{0, 7-4\} = 3$ and $\eth(7,t_1) = 1 = \eth(4,t_2)$. Thus

$$\eth(x * y, \min\{t_1,t_2\}) = \eth(3, 0.38) = 0 < 1 = \min\{\eth(7,t_1), \eth(4,t_2)\},$$

which shows that $\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

Let $\eth$ be a type-2 fuzzy set on X and $t \in [0,1]$. The *t-support* of the t-slice of $\eth$ is defined as the set of all elements of X with strictly positive membership degree under $\eth_t$, that is,

$$supp(\eth_t) = \{x \in X \mid \eth_t(x) > 0\}.$$

The *support* of $\eth$ is the set of all elements $x \in X$ for which there exists $t \in [0,1]$ such that $\eth(x,t) > 0$, that is,

$$supp(\eth) = \{x \in X \mid (\exists t \in [0,1])(\eth(x,t) > 0)\}.$$

The *t-core* of the t-slice of $\eth$ is defined to be the set of elements that have full membership degree under $\eth_t$, that is,

$$core(\eth_t) = \{x \in X \mid \eth_t(x) = 1\}.$$

The *core* of $\eth$ is the set of all elements $x \in X$ for which there exists $t \in [0,1]$ such that $\eth(x,t) = 1$, that is,

$$core(\eth) = \{x \in X \mid (\exists t \in [0,1])(\eth(x,t) = 1)\}.$$

It is clear that $core(\eth_t) \subseteq supp(\eth_t)$ for all $t \in [0,1]$, and $core(\eth) \subseteq supp(\eth)$.

Theorem 3.17. *Let $\eth$ be a type-2 fuzzy set on X and $t \in [0,1]$. If the t-slice of $\eth$ is a fuzzy subalgebra of $\mathcal{X}$, then its nonempty t-support $supp(\eth_t)$ and t-core $core(\eth_t)$ are subalgebras of $\mathcal{X}$.*

Proof. Let $(x,\mathfrak{a}), (y,\mathfrak{b}) \in supp(\eth_t) \times core(\eth_t)$. Then $\eth_t(x) > 0$, $\eth_t(y) > 0$, and $\eth_t(\mathfrak{a}) = 1 = \eth_t(\mathfrak{b})$. Thus $\eth_t(x * y) \geq \min\{\eth_t(x), \eth_t(y)\} > 0$ and

$$\eth_t(\mathfrak{a} * \mathfrak{b}) \geq \min\{\eth_t(\mathfrak{a}), \eth_t(\mathfrak{b})\} = 1$$

since $\eth_t$ is a fuzzy subalgebra of $\mathcal{X}$. Thus $x * y \in supp(\eth_t)$ and $\mathfrak{a} * \mathfrak{b} \in core(\eth_t)$. So $supp(\eth_t)$ and $core(\eth_t)$ are subalgebras of $\mathcal{X}$. □

The following example illustrates Theorem 3.17.

Example 3.18. Let $X = \{\varrho_0, \varrho_1, \varrho_2\}$ be a set in which the binary operation $*$ is given by Table 2. Then $\mathcal{X} := (X, *, \varrho_0)$ is a BCK-algebra (See [12]).

Define a type-2 fuzzy set $\eth$ in X as follows: for all $t \in [0,1]$, $\eth : X \times [0,1] \to [0,1]$ by $\eth(\varrho_0, t) = 1$, $\eth(\varrho_1, t) = t$ and $\eth(\varrho_2, t) = 1 - t$. Then $\eth_t$ is a fuzzy subalgebra of $\mathcal{X}$ and $\eth_t(\varrho_0) = 1$, $\eth_t(\varrho_1) = t$ and $\eth_t(\varrho_2) = 1 - t$. If $t = 0$, then $\eth_0(\varrho_0) = 1$, $\eth_0(\varrho_1) = 0$, and $\eth_0(\varrho_2) = 1$. Thus $supp(\eth_0) = \{\varrho_0, \varrho_2\}$ and $core(\eth_0) = \{\varrho_0, \varrho_2\}$ which are subalgebras of $\mathcal{X}$. If $t = 1$, then $\eth_1(\varrho_0) = 1$, $\eth_1(\varrho_1) = 1$ and $\eth_1(\varrho_2) = 0$. Thus $supp(\eth_1) = \{\varrho_0, \varrho_1\}$ and $core(\eth_0) = \{\varrho_0, \varrho_1\}$ which are subalgebras of $\mathcal{X}$. If $t \in (0,1)$, then $\eth_t(\varrho_0) = 1$, $\eth_t(\varrho_1) = t > 0$ and $\eth_t(\varrho_2) = 1 - t > 0$. Thus $supp(\eth_t) = \{\varrho_0, \varrho_1, \varrho_2\} = X$ and $core(\eth_0) = \{\varrho_0\}$ which are subalgebras of $\mathcal{X}$.

Table 2. $*$-table

$*$	ϱ_0	ϱ_1	ϱ_2
ϱ_0	ϱ_0	ϱ_0	ϱ_0
ϱ_1	ϱ_1	ϱ_0	ϱ_1
ϱ_2	ϱ_2	ϱ_2	ϱ_0

Theorem 3.19. *If a type-2 fuzzy set $\eth$ on X is a type-2 fuzzy subalgebra of $\mathcal{X}$, then its nonempty support and nonempty core are subalgebras of $\mathcal{X}$.*

Proof. Let $(x, \mathfrak{a}), (y, \mathfrak{b}) \in supp(\eth) \times core(\eth)$. Then there exist $(t_x, t_\mathfrak{a}), (t_y, t_\mathfrak{b}) \in [0,1] \times [0,1]$ with $\eth(x, t_x) > 0$, $\eth(y, t_y) > 0$, $\eth(\mathfrak{a}, t_\mathfrak{a}) = 1$ and $\eth(\mathfrak{b}, t_\mathfrak{b}) = 1$. Thus it follows from (3.1) that

$$\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\} > 0$$

and

$$\eth(\mathfrak{a} * \mathfrak{b}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\}) \geq \min\{\eth(\mathfrak{a}, t_\mathfrak{a}), \eth(\mathfrak{b}, t_\mathfrak{b})\} = 1.$$

So $(x * y, \mathfrak{a} * \mathfrak{b}) \in supp(\eth) \times core(\eth)$. Hence $supp(\eth)$ and $core(\eth)$ are subalgebras of $\mathcal{X}$. □

The following example shows that there exists a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$, which is not a type-2 fuzzy subalgebra but its nonempty support is a subalgebra of $\mathcal{X}$.

Example 3.20. Consider a BCK-algebra $\mathcal{X} := (X, *, \varrho_0)$, where $X = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ and the binary operation $*$ is given by Table 3.

Table 3. $*$-table

$*$	ϱ_0	ϱ_1	ϱ_2	ϱ_3	ϱ_4
ϱ_0	ϱ_0	ϱ_0	ϱ_0	ϱ_0	ϱ_0
ϱ_1	ϱ_1	ϱ_0	ϱ_1	ϱ_0	ϱ_0
ϱ_2	ϱ_2	ϱ_2	ϱ_0	ϱ_0	ϱ_2
ϱ_3	ϱ_3	ϱ_2	ϱ_1	ϱ_0	ϱ_2
ϱ_4	ϱ_4	ϱ_1	ϱ_4	ϱ_1	ϱ_0

Define a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$ by

$$\eth(x,t) = \begin{cases} 1 & \text{if } (x,t) = (\varrho_0, 1), \\ 0.8 & \text{if } (x,t) = (\varrho_3, 1), \\ 0.6 & \text{if } (x,t) = (\varrho_2, 1), \\ 0.3 & \text{if } (x,t) = (\varrho_1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Then $supp(\eth) = \{0, \varrho_1, \varrho_2, \varrho_3\}$ which is a subalgebra of $\mathcal{X}$. But for $t_1 = t_2 = 1$ we have

$$\begin{aligned}&\eth(\varrho_3 * \varrho_2, \min\{t_1, t_2\}) = \eth(\varrho_1, 1) = 0.3 < 0.6 = \min\{0.8, 0.6\}\\ &= \min\{\eth(\varrho_3, 1), \eth(\varrho_2, 1)\} = \min\{\eth(\varrho_3, t_1), \eth(\varrho_2, t_2)\}\end{aligned}$$

which shows that $\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

The following example shows that there exists a type-2 fuzzy set $\eth : X \times [0, 1] \to [0, 1]$, which is not a type-2 fuzzy subalgebra but its nonempty core is a subalgebra of $\mathcal{X}$.

Example 3.21. Consider the BCK-algebra $\mathcal{X} := (X, *, \varrho_0)$ in Example 3.20 and define a type-2 fuzzy set $\eth : X \times [0, 1] \to [0, 1]$ by

$$\eth(x, t) = \begin{cases} 1 & \text{if } (x, t) = (\varrho_1, 1),\\ 1 & \text{if } (x, t) = (\varrho_2, 1),\\ 1 & \text{if } (x, t) = (\varrho_3, 1),\\ 0.6 & \text{if } (x, t) = (\varrho_0, 1),\\ 1 & \text{if } (x, t) = (\varrho_0, 0.2),\\ 0.3 & \text{otherwise.}\end{cases}$$

Then $core(\eth) = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3\}$ which is a subalgebra of $\mathcal{X}$. For $t_1 = t_2 = 1$, we get

$$\begin{aligned}&\eth(\varrho_2 * \varrho_3, \min\{t_1, t_2\}) = \eth(\varrho_0, 1) = 0.6 < 1 = \min\{1, 1\}\\ &= \min\{\eth(\varrho_2, 1), \eth(\varrho_3, 1)\} = \min\{\eth(\varrho_2, t_1), \eth(\varrho_3, t_2)\}.\end{aligned}$$

Thus $\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

Given a type-2 fuzzy set $\eth$ on X and $\beta \in [0, 1]$, we consider the set

$$\eth^{\sup}_{\beta} := \{x \in X \mid \sup_{t\in[0,1]} \eth(x, t) \geq \beta\}$$

which is called the *super β-level set* of $\eth$. Also, the set

$$\eth^{\inf}_{\beta} := \{x \in X \mid \inf_{t\in[0,1]} \eth(x, t) \geq \beta\}$$

is called the *infer β-level set* of $\eth$.

Theorem 3.22. *If $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, then its nonempty super β-level set is a subalgebra of $\mathcal{X}$ for all $\beta \in [0, 1]$.*

Proof. Let $x, y \in \eth^{\sup}_{\beta}$ for every $\beta \in [0, 1]$. Then $t_x := \sup_{t\in[0,1]} \eth(x, t) \geq \beta$ and $t_y := \sup_{t\in[0,1]} \eth(y, t) \geq \beta$. Since $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, for every $(t_1, t_2) \in [0, 1] \times [0, 1]$ we have

$$\eth(x * y, \min\{t_1, t_2\}) \geq \min\{\eth(x, t_1), \eth(y, t_2)\}.$$

If we take the supremum over all t_1 and t_2, then

$$\sup_{t\in[0,1]} \eth(x * y, t) \geq \sup_{(t_1,t_2)\in[0,1]\times[0,1]} \min\{\eth(x, t_1), \eth(y, t_2)\}.$$

But $\eth(x,t_1)$ and $\eth(y,t_2)$ are bounded on $[0,1]$, we have the identity

$$\sup_{(t_1,t_2)\in[0,1]\times[0,1]} \min\{\eth(x,t_1),\eth(y,t_2)\} = \min\{\sup_{t_1\in[0,1]}\eth(x,t_1), \sup_{t_2\in[0,1]}\eth(y,t_2)\}.$$

It follows that

$$\sup_{t\in[0,1]}\eth(x*y,t) \geq \min\{\sup_{t_1\in[0,1]}\eth(x,t_1), \sup_{t_2\in[0,1]}\eth(y,t_2)\}.$$

Especially, we have

$$\sup_{t\in[0,1]}\eth(x*y,t) \geq \min\{\sup_{t\in[0,1]}\eth(x,t), \sup_{t\in[0,1]}\eth(y,t)\} = \min\{t_x,t_y\} \geq \beta.$$

Thus $x*y \in \eth_\beta^{\sup}$, so $\eth_\beta^{\sup}$ is a subalgebra of $\mathcal{X}$. □

The following example shows that the converse of Theorem 3.22 may not be true in general.

Example 3.23. Consider a BCK-algebra $\mathcal{X} := (X, *, \varrho_0)$, where $X = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ and the binary operation $*$ is given by Table 4.

Table 4. $*$-table

$*$	ϱ_0	ϱ_1	ϱ_2	ϱ_3	ϱ_4
ϱ_0	ϱ_0	ϱ_0	ϱ_0	ϱ_0	ϱ_0
ϱ_1	ϱ_1	ϱ_0	ϱ_0	ϱ_0	ϱ_0
ϱ_2	ϱ_2	ϱ_2	ϱ_0	ϱ_0	ϱ_0
ϱ_3	ϱ_3	ϱ_3	ϱ_3	ϱ_0	ϱ_0
ϱ_4	ϱ_4	ϱ_4	ϱ_3	ϱ_2	ϱ_0

Define a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$ by

$$\eth(\varrho_0,t) = 1 \text{ for all } t\in[0,1],$$

$$\eth(\varrho_1,t) = \begin{cases} 1 & \text{if } t = 0.4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eth(\varrho_2,t) = \begin{cases} 1 & \text{if } t = 0.3, \\ 0.62 & \text{if } t = 0.5, \\ 0.58 & \text{if } t = 0.7, \\ 0 & \text{otherwise.} \end{cases}$$

$$\eth(\varrho_3,t) = \begin{cases} 1 & \text{if } t = 0.8, \\ 0.75 & \text{if } t = 0.2, \\ 0.84 & \text{if } t = 0.9, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eth(\varrho_4,t) = \begin{cases} 1 & \text{if } t = 0.8, \\ 0.73 & \text{if } t = 0.5, \\ 0.39 & \text{if } t = 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

Table 5 clearly tabulates $\eth(x,t)$ for all $x\in X$ and some samples t, allowing a visual view of the counterexample.

TABLE 5. Type-2 fuzzy set $\eth(x,t)$

X	ϱ_0	ϱ_1	ϱ_2	ϱ_3	ϱ_4
$t = 0.2$	1	0	0	0.75	0
$t = 0.3$	1	0	1	0	0
$t = 0.4$	1	1	0	0	0
$t = 0.5$	1	0	0.62	0	0.73
$t = 0.6$	1	0	0	0	0.39
$t = 0.7$	1	0	0.58	0	0
$t = 0.8$	1	0	0	1	1
$t = 0.9$	1	0	0	0.84	0
otherwise	1	0	0	0	0

Then $\sup_{t\in[0,1]} \eth(x,t) = 1$ for all $x \in X$, so $\eth_\beta^{\sup} = X$ which is a subalgebra of $\mathcal{X}$ for all $\beta \in [0,1]$. But

$$\begin{aligned}&\eth(\varrho_4 * \varrho_3, \min\{0.5, 0.9\}) = \eth(\varrho_2, 0.5) = 0.62 < 0.73\\ &= \min\{0.73, 0.84\} = \min\{\eth(\varrho_4, 0.5), \eth(\varrho_3, 0.9)\}.\end{aligned}$$

Thus $\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

Theorem 3.24. *If $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, then its nonempty infer β-level set is a subalgebra of $\mathcal{X}$ for all $\beta \in [0,1]$.*

Proof. Let $\beta \in [0,1]$ be such that $\eth_\beta^{\inf} \neq \varnothing$ and let $x, y \in \eth_\beta^{\inf}$. Then $\inf_{t\in[0,1]} \eth(x,t) \geq \beta$ and $\inf_{t\in[0,1]} \eth(y,t) \geq \beta$. Since $\eth$ is a type-2 fuzzy subalgebra of $\mathcal{X}$, we have

$$\eth(x * y, \min\{t_x, t_y\}) \geq \min\{\eth(x, t_x), \eth(y, t_y)\}.$$

Thus we have

$$\begin{aligned}&\inf_{\min\{t_x,t_y\}\in[0,1]} \eth(x * y, \min\{t_x, t_y\}) \geq \inf_{(t_x,t_y)\in[0,1]\times[0,1]} \min\{\eth(x, t_x), \eth(y, t_y)\}\\ &= \min\{\inf_{t_x\in[0,1]} \eth(x, t_x), \inf_{t_y\in[0,1]} \eth(y, t_y)\}.\end{aligned}$$

So $\inf_{t\in[0,1]} \eth(x * y, t) \geq \min\{\inf_{t\in[0,1]} \eth(x,t), \inf_{t\in[0,1]} \eth(y,t)\} \geq \mathfrak{b}$ by taking $t = t_x = t_y$, that is, $x * y \in \eth_\beta^{\inf}$. Hence $\eth_\beta^{\inf}$ is a subalgebra of $\mathcal{X}$. □

The converse of Theorem 3.24 is not true in general as shown in the following example.

Example 3.25. Let $X = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3\}$ be a set in which the binary operation $*$ is given by Table 6. Then $\mathcal{X} := (X, *, \varrho_0)$ is a BCI-algebra (See [2, 12]). Define a type-2 fuzzy set $\eth : X \times [0,1] \to [0,1]$ by $\eth(\varrho_0, t) = 1$ for all $t \in [0,1]$, and for each element $z \in X$ with $z \neq \varrho_0$, take a fixed parameter $t_z \in [0,1]$ and set

$$\eth(z,t) = \begin{cases} 1 & \text{if } t = t_z, \\ 0 & \text{if } t \neq t_z. \end{cases}$$

Table 6. $*$-table

$*$	ϱ_0	ϱ_1	ϱ_2	ϱ_3
ϱ_0	ϱ_0	ϱ_0	ϱ_2	ϱ_2
ϱ_1	ϱ_1	ϱ_0	ϱ_3	ϱ_2
ϱ_2	ϱ_2	ϱ_2	ϱ_0	ϱ_0
ϱ_3	ϱ_3	ϱ_2	ϱ_1	ϱ_0

Then $\inf\limits_{t\in[0,1]} \eth(\varrho_0, t) = 1$ and $\inf\limits_{t\in[0,1]} \eth(z, t) = 0$. Thus If $\beta = 0$, then

$$\eth_\beta^{\inf} := \{x \in X \mid \inf_{t\in[0,1]} \eth(x, t) \geq \beta\} = X.$$

If $\beta \in (0, 1]$, then $\eth_\beta^{\inf} := \{x \in X \mid \inf\limits_{t\in[0,1]} \eth(x, t) \geq \beta\} = \{\varrho_0\}$. So $\eth_\beta^{\inf}$ is a subalgebra of $\mathcal{X}$ for all $\beta \in [0, 1]$. Taking $(t_x, t_y) \in [0, 1] \times [0, 1]$ forces $\eth(\varrho_1, t_x) = 1 = \eth(\varrho_2, t_y)$. Hence

$$\eth(\varrho_1 * \varrho_2, \min\{t_x, t_y\}) = \eth(\varrho_3, \min\{t_x, t_y\}) = 0$$

because $\eth(\varrho_3, t) = 1$ only at $t = t_{\varrho_3}$. Therefore

$$\eth(\varrho_1 * \varrho_2, \min\{t_x, t_y\}) = 0 < 1 = \min\{\eth(\varrho_1, t_x), \eth(\varrho_2, t_y)\}$$

which shows that $\eth$ is not a type-2 fuzzy subalgebra of $\mathcal{X}$.

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